1 Visibility Harmonics

We start off with the simple case of a point source at a distance a from the spin axis (which is assumed for now to be steady and at the center of the coordinate system). The polar coordinates of the visibility plane are (k, ϕ). The point-source visibility is:

$$V = A\ e^{ika\ cos\phi}$$
$$= A\ e^{ika\frac{z+z^{-1}}{2}}, \qquad \text{where } z = e^{i\phi}$$

$$= A\ [1\ +\ ika\frac{z+z^{-1}}{2}\ -\ \frac{(ka)^2}{2!}\ (\frac{z+z^{-1}}{2})^2 +\ \cdots]$$

$$= A\ \sum_{m=-\infty}^{\infty} J_m(ka)(iz)^m, \qquad \text{Abramowitz and Stegun, 1946}$$

Now we expand the exponential z into sines and cosines, and then write out the lowest-order terms explicitly:

$$V = A\ \sum_{m=-\infty}^{\infty} J_m(ka)\ i^m\ [cos(m\phi) + i\ sin(m\phi)], \qquad \phi = \phi - \psi$$

$$\begin{aligned} = A\ [&J_0(ka) + J_{-1}(ka)i^{-1}(cos\phi - isin\phi) +\ J_1(ka)i(cos\phi + isin\phi) + \\ &J_{-2}(ka)i^{-2}(cos2\phi - isin2\phi) +\ J_2(ka)i^2(cos2\phi + isin2\phi) + \\ &J_{-3}(ka)i^{-3}(cos3\phi - isin3\phi) +\ J_3(ka)i^3(cos3\phi + isin3\phi) + \\ &J_{-4}(ka)i^{-4}(cos4\phi - isin4\phi) +\ J_4(ka)i^4(cos4\phi + isin4\phi) + \cdots] \end{aligned}$$

$$\begin{aligned} = A\ [&J_0(ka) + iJ_1(ka)(cos\phi - isin\phi) +\ iJ_1(ka)(cos\phi + isin\phi) + \\ &-J_2(ka)(cos2\phi - isin2\phi) +\ -J_2(ka)(cos2\phi + isin2\phi) + \\ &-iJ_3(ka)(cos3\phi - isin3\phi) +\ -iJ_3(ka)(cos3\phi + isin3\phi) + \\ &J_4(ka)(cos4\phi - isin4\phi) +\ J_4(ka)(cos4\phi + isin4\phi) + \cdots] \end{aligned}$$

$$= A\ [J_0(ka) + 2iJ_1(ka)cos\phi - 2J_2(ka)cos2\phi - 2iJ_3(ka)cos3\phi + 2J_4(ka)cos4\phi + \cdots]$$

Now we can look at the Real and Imaginary parts of V separately, to see if there are any hidden relationships:

$$Re(V) = A\, cos(ka\; cos\phi) = A\, [J_0(ka) - 2J_2(ka)cos2\phi + 2J_4(ka)cos4\phi + \ldots]$$

$$Im(V) = A\; sin(ka\; cos\phi) = 2A\; [J_1(ka)cos\phi \;-\; J_3(ka)cos3\phi \;+\; \cdots]$$

The last two equations show that the coefficients of the Im(V) series must be related to the neighboring coeffcients of the Re(V) series because of the recurrence relation for Bessel functions.

$$J_m(z) = (z/2m)[J_{m-1}(z) + J_{m+1}(z)]$$

Since the visibility is a linear superposition of visibilities of point sources, it may be possible to exploit this recurrence formula to obtain the imaginary part of the visibility from the real part.
To clarify, we rewrite the equations in the form:

$$Re(V) = \sum_{-\infty}^{\infty} a_m cos(m\phi)$$

$$= A\, [\cdots + J_{-4}cos(-4\phi) - J_{-2}cos(-4\phi) + J_0 cos(0\phi) - J_2 cos(2\phi) + J_4 cos(4\phi) + \cdots]$$

$$Im(V) = \sum_{-\infty}^{\infty} b_m cos(m\phi)$$

$$A\, [\cdots - J_{-5}cos(-5\phi) + J_{-3}cos(-3\phi) - J_{-1}cos(-\phi) + J_1 cos(\phi) - J_3 cos(3\phi) + \cdots]$$

Comparing the coefficients with the recurrence relation, we find that:

$$b_m = (ka/2m) * (a_{m-1} - a_{m+1})$$

Although ka may be unknown, this equation shows that at least the profile of the imaginary coefficients b_m can be found from the a_m.

1.1 Source at Arbitary Azimuth

The case we just explored was very special: the source lay on the X axis, so there were no $sin(m\phi)$ terms in the Fourier expansions. Now we look at the case where the source is located at (a, ψ). The equations are very similar to the ones above, except that ϕ is replaced with $\phi - \psi$:

$$V = A\ e^{ika\ cos(\phi-\psi)}, \qquad z = e^{i(\phi-\psi)}$$

$$= A \sum_{m=-\infty}^{\infty} J_m(ka)\ i^m\ [cos(m(\phi-\psi) + i\ sin(m(\phi-\psi))]$$

So all the expansions remain the same with ϕ replaced by $\phi - \psi$
Now, however we come to a complication. Instead of a_m, we are given the Fourier coefficients $a_m e^{-im\psi}$ and $a_m e^{im\psi}$, where ψ is unknown:

$$Re(V) = \sum_{-\infty}^{\infty} a_m cos(m(\phi-\psi)) = \frac{1}{2}\sum_{-\infty}^{\infty}\{a_m e^{-im\psi}\ e^{im\phi} + a_m e^{im\psi}\ e^{-im\phi}\}$$

This would be a trivial problem, since the product of the two terms is a real positive number, a_m^2, except that the sign of a_m must be determined and ψ must be independent of m. If we force a_m to be positive, then we put the onus for the sign on $e^{-im\psi}$, and this can make ψ be different for different m, which is not allowed.

Once the a_m and ψ have been found, the same recurrence relations hold, and one may determine Im(V) from Re(V).

1.2 Superposed Sources

Some tests using IDL have been made with double sources, and it appears that the recurrence relation that relates the real and imaginary parts still works to a good approximation.